

Deformations of three-dimensional metrics

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Abstract We examine three-dimensional metric deformations based on a tetrad transformation through the action the matrices of scalar field. We describe by this approach to deformation the results obtained by Coll et al. in [1], where it is stated that any three-dimensional metric was locally obtained as a deformation of a constant curvature metric parameterized by a 2-form. To this aim, we construct the corresponding deforming matrices and provide their classification according to the properties of the scalar σ and of the vector \mathbf{s} used in [1] to deform the initial metric. The resulting causal structure of the deformed geometries is examined, too. Finally we apply our results to a spherically symmetric three geometry and to a space sector of Kerr metric.

Keywords space-time deformations, scalar fields, three-dimensional metrics, conformal methods

1 Introduction

Differently from other classical field theories there is not a general solution of the Einstein equations. Frequently it is necessary to impose some symmetries on space-time to find exact solutions. Despite the restrictions introduced considering the symmetries, exact solutions are important in general relativity, as well as in extended gravity, because they provide a deep insight of the gravitational issues. Many general properties of the gravitational field would not have been well understood without knowing the exact solutions of general

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relativity. Important examples are the physics of black holes and development of the cosmological models. In addition by comparing the solutions of general relativity with the corresponding solutions the extended gravity theories has allowed us to understand the fundamental differences between the various theories and to formulate new scenarios in a cosmological context in addressing the dark matter and dark energy problems.

However real space-time never shows a perfect symmetry. Deviations from symmetry may be small, but still significative. Just to give an example if we want to make good geodesy on the earth it is necessary to consider the earth as a geoid, whose deviations from being an exact ellipsoid are $\delta R/R \sim 10^{-5}$, where R is the average radius of the earth. Similarly the deviations of the cosmic gravitational background temperature are of the order of $\delta T/T \sim 10^{-5}$, which means that trying to introduce a more accurate metric is a problem that can have a certain interest in the description of the universe, especially in the era of precision cosmology. How to fit an idealized exactly homogeneous and isotropic solution of the Einstein equations to a lumpy universe was considered in [2] by examining averaging, normal coordinates and null data and trying to relate them to data coming from astronomical observations.

Along with the exact solutions we must also consider the solutions obtained by slightly modifying the space-time background. For example gravitational waves were found by considering the propagation of small deviations from the Minkowski background. In the cosmological models. The perturbations introduced into the bottom of homogeneous and isotropic in fact lead us to consider new solutions of Einstein's equations. But these solutions are not completely covariant in the spirit of Einstein's theory as perturbations that are small in a particular coordinate system can be made arbitrarily large with particular changes of coordinates.

We can deal with these problems in general by introducing the concept of deformation of a space-time metric, covariant way with respect to the coordinate transformations. By deformations we mean any kind of transformation of the metric which modifies the geometrical properties of the original space-time. In other words a deformation is an application defined on the superspace which associates to each metric (modulo diffeomorphisms) another metric (modulo diffeomorphisms).

Deformations have been widely considered in the literature. Conformal transformations are a first example of metric deformations. They have the important property that they preserve the causal structure of space-time. For recent literature in the subject see for example [3, 4, 5, 6, 7, 8]. The metric deformations so defined provide a valuable method for the investigation of some local and global properties of space-times. For example in the investigation of stability of space-times with stationary singularities [3, 4, 5, 6, 7, 8, 9, 10, 11]. Noteworthy the solutions of the $f(R)$ theories and the corresponding solutions of Einstein equations are related by a conformal transformation.

Another way to obtain a deformed manifold is by considering the Ricci flow, i.e. a flow on the space of metrics (in different space-time dimensions) described by a parabolic equation. It makes sense for compact manifolds and

it describes the expansion of negatively curved regions of the manifold and the contraction of positively curved regions. Imposing the appropriate corrections it can describe the deformation of a manifold to a smooth manifold. Other similar flows have been introduced in literature like the Yamabe flow and the Calabi flow. Ricci flows has been used to discuss a smoothing procedure deforming a family of inhomogeneous cosmological models into a FLRW universe which describes an average behavior of the more realistic universe [12]. On the other hand perturbations in a homogeneous and isotropic universe lead to a deformation of the FLRW metric (see for example [13]). A relevant example of deformation of a metric to another was given by Newman and Janis in [14], where they showed that the Kerr metric can be obtained by a simple complex transformation of the Schwarzschild metric. The same algorithm was used to find the Kerr-Newman metric. The construction of the BTZ solution for a black hole in [15] may be considered as a deformation of the AdS metric. Finally we mention that deformation of a metric by introducing torsion was studied in [11].

It seems necessary to give a general description of a metric deformations. A first attempt was given by the work of Coll and collaborators [1] and [16, 17, 18, 19, 20, 21], where a particular definition was given to the deformations of a space-time metric. They showed how a generic metric can be related to a metric with constant curvature and tried to extend this results to any dimension.

In [22, 23] space-time metric deformations were introduced in a more general way. Such deformations are defined in Sec. (2), they are based on the introduction of matrices of scalar fields which transform a tetrad taking a linear combination of the vectors (or respectively convectors) at each point of the space-time. It results that we find a new metric by reconstructing it with the new tetrad. In this paper we study the deformations of a 3-metric, as defined in [22, 23], in relation with the work of Coll and collaborators. As a result we find a broad classification of metric deformations and we analyze the consequences to the casual structure of the deformed manifold. Our results are covariant with respect to coordinate transformations, even if the price payed for this is that the matrices introduced are well-defined except for a left Lorentz transformation, but such gauge freedom disappears after constructing the deformed metric.

This article is organized as follows: in Sec. (2) we briefly introduce our definition of space-time deformations. In Section (3) we will consider deformations of three-dimensional metrics, we first present a general procedure outlined in [16]. Deforming matrices for three-dimensional metrics are then discussed in Sec. (4), exploring in particular the real solutions in Sec. (4.1), and the complex solutions in Sec. (4.2). In Sec. (5) we investigate the casual structure of a deformed manifold compared with the original one and finally in Sec. (6) we shall consider the deformations of some three-dimensional Riemannian manifolds into flat metrics, applying the results to the metric of a

the three-dimensional sphere in Sec. (6.1) and to the three-dimensional Kerr space in Sec. (6.2). In Section (7) some concluding remarks are presented.

2 Deformation of metrics

As stated in the introduction in order to deform a metric we look for an application defined on the superspace which associates to each metric (modulo diffeomorphisms) another metric (modulo diffeomorphisms).

One way to deform a metric is the following, let us consider the decomposition of a covariant metric g on an n -dimensional manifold M in terms of a cotetrad field ω_a^A and correspondingly the decomposition of the contravariant metric in terms of the tetrad field e_A^a (such that $\omega_a^A e_A^a = \delta_B^A$)

$$g_{ab} = \eta_{AB} \omega_a^A \omega_b^B, \quad g^{ab} = \eta^{AB} e_A^a e_B^b, \quad (1)$$

where η_{AB} is the Minkowski metric in tetrad indices¹. Then, given a matrix ϕ_B^A of scalar fields defined on M , which is different from a Lorentz matrix at least in one point, we obtain a covariant tensor \tilde{g}_{ab}

$$\tilde{g}_{ab} \equiv \eta_{AB} \phi_C^A \phi_D^B \omega_a^C \omega_b^D, \quad (2)$$

which can be considered as deformed metric in a different manifold \tilde{M} together with the frames defined by the deformed cotetrad field and $\tilde{\omega}_a^B$ and tetrad field \tilde{e}_A^a defined by the compositions

$$\tilde{\omega}_a^B \equiv \omega_a^A \phi_A^B, \quad \tilde{e}_A^a \equiv \psi_A^B e_B^a, \quad (3)$$

where ψ_A^B is a second matrix of scalar fields. We shall consider in the following that ϕ and ψ satisfy the condition

$$\psi_B^A = \phi_B^{-1A}. \quad (4)$$

so that the following relations hold

$$\tilde{\omega}_a^A \tilde{e}_B^a = \delta_B^A, \quad \tilde{\omega}_a^A \tilde{e}_A^b = \delta_a^b. \quad (5)$$

Thus, the metric \tilde{g}_{ab} and the contravariant tensor \tilde{g}^{ab} is defined respectively by (2) as

$$\tilde{g}_{ab} = \eta_{AB} \phi_C^A \phi_D^B \omega_a^C \omega_b^D, \quad \tilde{g}^{ab} = \eta^{AB} \phi^{-1A}_C \phi^{-1B}_D e_C^a e_D^b, \quad (6)$$

or

$$\tilde{g}_{ab} = \eta_{AB} \tilde{\omega}_a^A \tilde{\omega}_b^B, \quad \tilde{g}^{ab} = \eta^{AB} \tilde{e}_A^a \tilde{e}_B^b \quad (7)$$

On the other hand the previous relations can be also read as deformations of the Minkowski metrics η_{AB} and η^{AB} by a second couple of symmetric matrices of scalar fields, \mathcal{G}_{CD} and \mathcal{G}^{CD} defined on the manifold M respectively as

$$\mathcal{G}_{CD} \equiv \eta_{AB} \phi_C^A \phi_D^B, \quad \text{and} \quad \mathcal{G}^{CD} \equiv \eta_{AB} \psi_A^C \psi_B^D \quad (8)$$

¹ Capital latin letters A are tensorial indices while Latin letter a denote the tensorial character of each object i.e. they are spacetime indices.

and then writing the metrics \tilde{g} as

$$\tilde{g}_{ab} = \mathcal{G}_{CD} \omega_a^C \omega_b^D, \quad \tilde{g}^{ab} = \mathcal{G}^{CD} e_C^a e_D^b. \quad (9)$$

We do not impose particular requirement on the matrices $\phi_{\text{B}}^{\text{A}}$ except they have to be invertible. They may even be singular in some point, for example by deforming a Minkowski space-time to a Schwarzschild space-time, inevitably we must take a singular matrix.

Finally we note that since the Minkowski metric is invariant by Lorentz transformations, that

$$\tilde{g}_{ab} = \eta_{AB} \phi_C^A \phi_D^B \omega_a^C \omega_b^D = \eta_{AB} \Lambda_D^A \phi_C^D \Lambda_F^B \phi_D^F \omega_a^C \omega_b^D, \quad (10)$$

where the matrices $\Lambda_{\text{B}}^{\text{A}}$ which are Lorentz at each point of the manifold. Then the deformation matrices are not uniquely defined and to each metric deformation we have to associate a family of matrices.

Particular simple deformations are the deformations form $\phi_{\text{C}}^{\text{A}}(x) = \Omega(x) \delta_{\text{C}}^{\text{A}}$, and more generally $\phi_{\text{C}}^{\text{A}}(x) = \Omega(x) \Lambda_{\text{C}}^{\text{A}}$, which generate a conformal transformation of the metric with conformal factor $\Omega^2(x)$.

3 Deformations of three-dimensional metrics

In the context of deformations of three-dimensional metrics Coll et al. [1] proved the following theorem:

Theorem 1 *Let M be a three-dimensional Riemannian manifold, and g a generic 3-dimensional metric. Then g may be locally obtained from a constant curvature metric h by a deformation of the form*

$$g = \sigma h + \epsilon \mathbf{s} \otimes \mathbf{s}, \quad (11)$$

where σ and \mathbf{s} are respectively a scalar function and a differential 1-form on M and $\epsilon = \pm 1$.

Let us note that, fixed g and h , σ and \mathbf{s} are not independent as, by Riemann's theorem, a generic n -dimensional metric has $f = n(n-1)/2$ degrees of freedom. Thus, according to it, a scalar relation $\Psi(\sigma, \|\mathbf{s}\|) = 0$ between σ and \mathbf{s} has to be imposed and then the metric can be defined, at most, by three independent functions. On the other hand the direction \mathbf{s} is not uniquely determined by the deformed metric g so that relation (11) can be achieved in an infinite number of ways.

Moreover, **Theorem 1** establishes also the following:

Theorem 2 *Let (M, g) be a Riemannian three-manifold, there locally exist a function ϕ and a 1-form μ such that the tensor:*

$$\tilde{g} := \phi g - \epsilon \mu \otimes \mu \quad (12)$$

with $\epsilon = \pm 1$ is Riemannian metric with constant curvature; It is possible then add an arbitrary relation among the function ϕ and $|\mu|^2 := g^{ij} \mu_i \mu_j$.

The inverse metric is \tilde{g}_{ij}

$$h^{rj} \equiv \phi^{-1} \left(g^{rj} + \frac{1}{\phi - m_0} M^{rj} \right),$$

where $M_{ij} \equiv \epsilon \mu_i \mu_j$ and $m_0 \equiv g^{ij} M_{ij} = \epsilon |\mu|^2$. This result follows directly from on substituting \tilde{g} with h , ϕ^{-1} with σ and $\phi^{-1/2} \mu$ with \mathbf{s} . Theorem 1 and 2 before have been proved in [1] in the case of three-dimensional Riemannian manifold. Nevertheless Llosa and Soler extended the results in [16] to a semi-Riemannian n -dimensional manifold: having in mind that any semi-Riemannian metric in a n -manifold has $n(n-1)/2$ degrees of freedom, as many as the number of component of a differential 2-form, they proved that this metric can be obtained as a deformation of a constant curvature metric, this deformations being parameterized by a two form [16]. It is interesting to note that the Kerr-Schild class of metrics in general relativity could be seen as particular case of the relation (11) where \mathbf{s} is a null vector and h is the Minkowski metric. On the other hand it is well known that, according to the Gauss theorem, any two-dimensional semi-Riemannian metric can be always locally mapped by a conformal transformation into Minkowski spacetime, then it is easy to see that Eq. (11) could be used as a map from three-dimensional spaces into spaces of constant curvature, and in this sense the result (11) generalizes the Gauss theorem in three-dimensions. Since the conformal factor σ can be written as a function of the \mathbf{s} , the deformation (11) establishes also, at fixed h , a relation between the set of metrics g and that of \mathbf{s} . However we stress that the (11) is valid only locally (in the neighborhood of the Cauchy surface) and moreover the proof of Theorem (1) is valid only in the analytical case where all the data are real analytic functions. The law (12), can be seen as a deformation of a metric g by a vector μ to obtain a constant metric \tilde{g} , has not a unique solution but, a part the freedom of the choice of the constraint Ψ , a great arbitrariness is left also in the choice of the Cauchy data (see [16]). As pointed out in [16] and as it will be discussed in the following analysis, as a consequence of this arbitrariness there is a great variety of vectors \mathbf{s} that deform a given constant curvature metric into another metric which has the same constant curvature.

4 Deforming matrices for three-dimensional metrics

In this section we focus on Theorem 1 looking for the deforming matrices associated with the transformation (11), in particular we find the explicit expression for the generic element of the deforming matrix associated to (11), distinguishing the complex scalar fields from the real ones. First we remark that Theorem 1 is proved locally in the analytical case and moreover it defines a constant curvature Riemannian metric. Deformation (6) relates two metric tensors by means of generic, non necessarily real, scalar fields; this definition is apparently universal, in the sense that it is able to be applied in any manifold M by means of a general choice of the triad and the scalar fields defined on

M . Hence, here we restrict ourselves to the case of constant curvature metric. In particular we prove that, since Eq. (11) can be recovered from (6), this can be regarded as a particular case of deformation (7). Let us write the metrics h and g in the (11) in the triad of covectors ω_b^D , as:

$$h_{ab} = \eta_{AB} \omega_a^A \omega_b^B, \quad g_{ab} = \mathcal{G}_{CD} \omega_a^C \omega_b^D \quad (13)$$

where \mathcal{G}_{CD} is a second deforming matrix. Equation (13) can be written in terms of the deforming matrix ϕ_C^A as follows

$$g_{ab} = \eta_{AB} \phi_C^A \phi_D^B \omega_a^C \omega_b^D \quad (14)$$

Thus, comparing Eq.(14) with the Eq.(11), we infer that the following identity should always hold

$$g_{ab} = \sigma h_{ab} + \epsilon s_a s_b = \eta_{AB} \phi_C^A \phi_D^B \omega_a^C \omega_b^D. \quad (15)$$

In order to solve Eq.(15) respect to the variable ϕ_C^A , we consider the fields σ and \mathbf{s} as known terms; thus the solution will give the field ϕ_C^A in (14) as a function of σ and \mathbf{s} . It is important to note that in the following analysis we don't fix the gauge condition as a scalar relation $\Psi(\sigma, \|\mathbf{s}\|) = 0$ between σ and \mathbf{s} ; nevertheless, it should be stressed that the deforming matrices are not uniquely defined but, since they form a right coset for right composition with Lorentz matrices Λ_C^A , every element we are going to obtain from the Eq.(15) identifies all the matrices of its equivalent class. We therefore consider a part the case of real solutions in Sec. (4.1), and the complex scalar field case in Sec. (4.2).

4.1 Real solutions

In this first case we study the real solutions of Eq.(14). In order to do this, we consider the following expression for the real scalar field ϕ_C^A :

$$\phi_C^A = \sqrt{\sigma} \delta_C^A + \alpha s^A s_C, \quad (16)$$

where $\sigma > 0$ and α is a scalar field. Therefore the second deforming matrix is:

$$\mathcal{G}_{CD} = \eta_{AB} (\sqrt{\sigma} \delta_C^A + \alpha s^A s_C) (\sqrt{\sigma} \delta_D^B + \alpha s^B s_D), \quad (17)$$

thus, we obtain

$$\begin{aligned} \mathcal{G}_{CD} = & \eta_{AB} \sqrt{\sigma} \sqrt{\sigma} \delta_C^A \delta_D^B + \eta_{AB} \sqrt{\sigma} \delta_C^A \alpha s^B s_D + \\ & + \eta_{AB} \alpha s^A s_C \sqrt{\sigma} \delta_D^B + \eta_{AB} \alpha^2 s^A s_C s^B s_D. \end{aligned} \quad (18)$$

or also

$$\mathcal{G}_{CD} = \sigma \eta_{CD} + 2\alpha \sqrt{\sigma} s_C s_D + \alpha^2 \eta_{AB} s^A s^B s_D s_C. \quad (19)$$

Collecting the terms in (19), we have:

$$\mathcal{G}_{CD} = \sigma \eta_{CD} + [2\alpha \sqrt{\sigma} + \alpha^2 (\eta_{AB} s^A s^B)] s_C s_D. \quad (20)$$

Finally, comparing Eq. (20) with Eq.(15), we obtain the following equation:

$$[2\alpha\sqrt{\sigma} + \alpha^2(\eta_{AB}s^A s^B) - \epsilon]s_C s_D = 0, \quad (21)$$

consequently

$$2\alpha\sqrt{\sigma} + \alpha^2(\eta_{AB}s^A s^B) - \epsilon = 0. \quad (22)$$

By setting $\|\mathbf{s}\|^2 \equiv \eta_{AB}s^A s^B$, Eq.(22) reduces to:

$$\|\mathbf{s}\|^2 \alpha^2 + 2\sqrt{\sigma}\alpha - \epsilon = 0. \quad (23)$$

We can read Eq.(23) as a second order equation respect to the variable α , with $\epsilon = \pm 1$.

Therefore, we have the two following possibilities:

$$1. \quad \|\mathbf{s}\|^2 \alpha^2 + 2\sqrt{\sigma}\alpha - 1 = 0 \quad \text{for} \quad \epsilon = +1 \quad (24)$$

$$2. \quad \|\mathbf{s}\|^2 \alpha^2 + 2\sqrt{\sigma}\alpha + 1 = 0 \quad \text{for} \quad \epsilon = -1 \quad (25)$$

In the first case the discriminant $\Delta/4$ of the equation (24) is:

$$\frac{\Delta}{4} = \sigma + \|\mathbf{s}\|^2. \quad (26)$$

Then we discuss the case $\Delta/4 > 0$ in Sec. (4.1.1) and solutions for $\Delta/4 = 0$ in Sec. (4.1.2).

4.1.1 Case: $\Delta/4 > 0$

This first case occurs when:

$$\|\mathbf{s}\|^2 > -\sigma, \quad (\epsilon = +1),$$

In particular, if $\|\mathbf{s}\|^2 = 0$ the quantity $\Delta/4$ (26) is strictly positive, where $\sigma \neq 0$ and the solution of the first order equation Eq.(23) is: $\alpha = 1/2\sqrt{\sigma}$, and ϕ_C^A is in this case:

$$\phi_C^A = \sqrt{\sigma}\delta_C^A + \frac{1}{2\sqrt{\sigma}}s^A s_C. \quad (27)$$

On the other hand for $\|\mathbf{s}\| \neq 0$ we have two real solutions of Eq.(24), α_{\mp} respectively:

$$\alpha_{\mp} = \frac{-\sqrt{\sigma} \mp \sqrt{\sigma + \|\mathbf{s}\|^2}}{\|\mathbf{s}\|^2}; \quad (28)$$

therefore the first deforming matrix ϕ_C^A is

$$\phi_C^A = \sqrt{\sigma}\delta_C^A + \frac{-\sqrt{\sigma} \mp \sqrt{\sigma + \|\mathbf{s}\|^2}}{\|\mathbf{s}\|^2}s^A s_C. \quad (29)$$

4.1.2 Case: $\Delta/4 = 0$

This case holds when $\|\mathbf{s}\|^2 = -\sigma$, therefore the solution of Eq.(24) is $\alpha = 1/\sqrt{\sigma}$. In this case the deforming matrix is :

$$\phi_{\mathbf{C}}^{\mathbf{A}} = \sqrt{\sigma}\delta_{\mathbf{C}}^{\mathbf{A}} + \frac{1}{\sqrt{\sigma}}\mathbf{s}^{\mathbf{A}}\mathbf{s}_{\mathbf{C}}. \quad (30)$$

Finally, the discriminant of Eq.(25)

$$\|\mathbf{s}\|^2\alpha^2 + 2\sqrt{\sigma}\alpha + 1 = 0 \quad (\epsilon = -1)$$

is

$$\frac{\Delta}{4} = \sigma - \|\mathbf{s}\|^2. \quad (31)$$

Therefore, two subcases occur for $\Delta/4 > 0$ in Sec. (4.1.3) and $\Delta/4 = 0$ in Sec. (4.1.4).

4.1.3 Case: $\Delta/4 > 0$

This case holds for $\|\mathbf{s}\|^2 < \sigma$ ($\epsilon = -1$). In particular, if $\|\mathbf{s}\|^2 = 0$ the discriminant (31) is strictly positive, and the solution of the first order equation Eq.(25) is:

$$\alpha = -\frac{1}{2\sqrt{\sigma}}$$

and $\phi_{\mathbf{C}}^{\mathbf{A}}$ is:

$$\phi_{\mathbf{C}}^{\mathbf{A}} = \sqrt{\sigma}\delta_{\mathbf{C}}^{\mathbf{A}} - \frac{1}{2\sqrt{\sigma}}\mathbf{s}^{\mathbf{A}}\mathbf{s}_{\mathbf{C}}. \quad (32)$$

On the other hand for $\|\mathbf{s}\| \neq 0$ there are two real solutions of Eq.(25), α_{\mp} :

$$\alpha_{\mp} = \frac{-\sqrt{\sigma} \mp \sqrt{\sigma - \|\mathbf{s}\|^2}}{\|\mathbf{s}\|^2}. \quad (33)$$

The deforming matrix in this case is:

$$\phi_{\mathbf{C}}^{\mathbf{A}} = \sqrt{\sigma}\delta_{\mathbf{C}}^{\mathbf{A}} + \frac{-\sqrt{\sigma} \mp \sqrt{\sigma - \|\mathbf{s}\|^2}}{\|\mathbf{s}\|^2}\mathbf{s}^{\mathbf{A}}\mathbf{s}_{\mathbf{C}}. \quad (34)$$

4.1.4 Case : $\Delta/4 = 0$

This case holds if

$$\|\mathbf{s}\|^2 = \sigma. \quad (35)$$

Thus, we find the following solution:

$$\alpha = -\frac{1}{\sqrt{\sigma}}. \quad (36)$$

in this case the deforming matrix is:

$$\phi_C^A = \sqrt{\sigma} \delta_C^A - \frac{1}{\sqrt{\sigma}} s^A s_C. \quad (37)$$

Finally, as shown in Table 1, the solutions for $\epsilon = \pm 1$ exist together for $\|s\| \in [-\sigma, \sigma]$. On the contrary, to $\|s\| < -\sigma$ and to $\|s\| > \sigma$ correspond the solutions $\epsilon = \mp 1$ respectively.

	$\epsilon = -1$	$\epsilon = +1$
$\ s\ ^2 < -\sigma$	$\phi_C^A = \sqrt{\sigma} \delta_C^A + \frac{-\sqrt{\sigma} \mp \sqrt{\sigma - \ s\ ^2}}{\ s\ ^2} s^A s_C$	✓
$\ s\ ^2 = -\sigma$	$\phi_C^A = \sqrt{\sigma} \delta_C^A + \frac{1 \pm \sqrt{2}}{\sqrt{\sigma}} s^A s_C$	$\phi_C^A = \sqrt{\sigma} \delta_C^A + \frac{1}{\sqrt{\sigma}} s^A s_C$
$\ s\ ^2 \in]-\sigma, \sigma[$	$\phi_C^A = \sqrt{\sigma} \delta_C^A + \frac{-\sqrt{\sigma} \mp \sqrt{\sigma - \ s\ ^2}}{\ s\ ^2} s^A s_C$	$\phi_C^A = \sqrt{\sigma} \delta_C^A + \frac{-\sqrt{\sigma} \mp \sqrt{\sigma + \ s\ ^2}}{\ s\ ^2} s^A s_C$
$\ s\ ^2 = 0$	$\phi_C^A = \sqrt{\sigma} \delta_C^A - \frac{1}{2\sqrt{\sigma}} s^A s_C$	$\phi_C^A = \sqrt{\sigma} \delta_C^A + \frac{1}{2\sqrt{\sigma}} s^A s_C$
$\ s\ ^2 = \sigma$	$\phi_C^A = \sqrt{\sigma} \delta_C^A - \frac{1}{\sqrt{\sigma}} s^A s_C$	$\phi_C^A = \sqrt{\sigma} \delta_C^A + \frac{-1 \pm \sqrt{2}}{\sqrt{\sigma}} s^A s_C$
$\ s\ ^2 > \sigma$	✓	$\phi_C^A = \sqrt{\sigma} \delta_C^A + \frac{-\sqrt{\sigma} \mp \sqrt{\sigma + \ s\ ^2}}{\ s\ ^2} s^A s_C$

Table 1 Deforming matrices for three-dimensional metrics. 1° case: Real solutions.

4.2 Complex solutions

In this section we study the complex solutions of Eq. (14). Thus, we write the scalar field ϕ_C^A as:

$$\phi_C^A = \sqrt{\sigma} \delta_C^A + \alpha s^A s_C, \quad (38)$$

where now α and $\sqrt{\sigma}$ are two complex scalar fields. Therefore in (38) we split the real part of scalar fields from the imaginary one. We can write:

$$\phi_C^A = (\text{Re } \sqrt{\sigma} + \imath \text{Im } \sqrt{\sigma}) \delta_C^A + (\text{Re } \alpha + \imath \text{Im } \alpha) s^A s_C, \quad (39)$$

or also:

$$\phi_C^A = (\text{Re } \sqrt{\sigma}) \delta_C^A + (\text{Re } \alpha) s^A s_C + \imath (\text{Im } \sqrt{\sigma} \delta_C^A + \text{Im } \alpha s^A s_C). \quad (40)$$

We consider the case $\sigma > 0$ in Sec. (4.2.1) and $\sigma < 0$ in Sec. (4.2.2)

4.2.1 Case: $\sigma > 0$

In this first case $\sqrt{\sigma} \equiv \text{Re } \sqrt{\sigma}$ and the expression (40) becomes:

$$\phi_{\text{C}}^{\text{A}} = \sqrt{\sigma} \delta_{\text{C}}^{\text{A}} + \text{Re } \alpha s^{\text{A}} s_{\text{C}} + i \text{Im } \alpha s^{\text{A}} s_{\text{C}}, \quad (41)$$

therefore the second deforming matrix \mathcal{G}_{CD} is:

$$\begin{aligned} \mathcal{G}_{\text{CD}} = \eta_{\text{AB}} & \left(\sqrt{\sigma} \delta_{\text{C}}^{\text{A}} + \text{Re } \alpha s^{\text{A}} s_{\text{C}} + i \text{Im } \alpha s^{\text{A}} s_{\text{C}} \right) \times \\ & \left(\sqrt{\sigma} \delta_{\text{D}}^{\text{B}} + \text{Re } \alpha s^{\text{B}} s_{\text{D}} + i \text{Im } \alpha s^{\text{B}} s_{\text{D}} \right). \end{aligned} \quad (42)$$

In particular for $\|\mathbf{s}\| = 0$ we obtain, from Eq.(42) the following real solutions $\phi_{\text{C}}^{\text{A}}$:

$$\phi_{\text{C}}^{\text{A}} = \sqrt{\sigma} \delta_{\text{C}}^{\text{A}} + \epsilon \frac{1}{2\sqrt{\sigma}} s^{\text{A}} s_{\text{C}} \quad (43)$$

Finally, comparing Eq.(42) with Eq.(15) we obtain the following expressions for α in the case $\|\mathbf{s}\| \neq 0$:

$$\alpha_{\mp} = -\frac{\sqrt{\sigma}}{\|\mathbf{s}\|^2} \mp i \frac{1}{\|\mathbf{s}\|^2} \sqrt{-(\sigma + \epsilon \|\mathbf{s}\|^2)} \quad \text{for } \epsilon \|\mathbf{s}\|^2 \leq \sigma, \quad (44)$$

or,

$$\alpha_{\mp} = -\frac{\sqrt{\sigma}}{\|\mathbf{s}\|^2} \mp i \frac{1}{\|\mathbf{s}\|^2} \sqrt{-(\sigma - \|\mathbf{s}\|^2)}, \quad \text{for } \epsilon = -1 \quad \text{and} \quad \|\mathbf{s}\|^2 \geq \sigma. \quad (45a)$$

$$\alpha_{\mp} = -\frac{\sqrt{\sigma}}{\|\mathbf{s}\|^2} \mp i \frac{1}{\|\mathbf{s}\|^2} \sqrt{-(\sigma + \|\mathbf{s}\|^2)}, \quad \text{for } \epsilon = +1 \quad \text{and} \quad \|\mathbf{s}\|^2 \leq -\sigma. \quad (45b)$$

Using the Eq. (45a) in (41), we obtain for the field $\phi_{\text{C}}^{\text{A}}$

$$\begin{aligned} \phi_{\text{C}}^{\text{A}} = \sqrt{\sigma} \delta_{\text{C}}^{\text{A}} - \frac{\sqrt{\sigma}}{\|\mathbf{s}\|^2} s^{\text{A}} s_{\text{C}} \mp i \frac{1}{\|\mathbf{s}\|^2} \sqrt{-(\sigma - \|\mathbf{s}\|^2)} s^{\text{A}} s_{\text{C}} \\ \text{for } \epsilon = -1, \quad \text{and} \quad \|\mathbf{s}\|^2 \geq \sigma. \end{aligned} \quad (46)$$

In particular we find the solution

$$\phi_{\text{C}}^{\text{A}} = \sqrt{\sigma} \delta_{\text{C}}^{\text{A}} - \frac{1}{\sqrt{\sigma}} s^{\text{A}} s_{\text{C}} \quad \text{for } \epsilon = -1, \quad \|\mathbf{s}\|^2 = \sigma. \quad (47)$$

While, using the expression (45b) in (41), we obtain for the field $\phi_{\text{C}}^{\text{A}}$:

$$\begin{aligned} \phi_{\text{C}}^{\text{A}} = \sqrt{\sigma} \delta_{\text{C}}^{\text{A}} - \frac{\sqrt{\sigma}}{\|\mathbf{s}\|^2} s^{\text{A}} s_{\text{C}} \mp i \frac{1}{\|\mathbf{s}\|^2} \sqrt{-(\sigma + \|\mathbf{s}\|^2)} s^{\text{A}} s_{\text{C}}, \\ \text{for } \epsilon = +1 \quad \text{and} \quad \|\mathbf{s}\|^2 \leq -\sigma, \end{aligned} \quad (48)$$

where in particular

$$\phi_{\text{C}}^{\text{A}} = \sqrt{\sigma} \delta_{\text{C}}^{\text{A}} + \frac{1}{\sqrt{\sigma}} s^{\text{A}} s_{\text{C}} \quad \text{for } \epsilon = +1, \quad \|\mathbf{s}\|^2 = -\sigma. \quad (49)$$

4.2.2 Case: $\sigma < 0$

In this second case we write the second deforming matrix \mathcal{G}_{CD} as:

$$\mathcal{G}_{\text{CD}} = \eta_{\text{AB}} \left[\text{Re} \sqrt{\sigma} \delta_{\text{C}}^{\text{A}} + \text{Re} \alpha s^{\text{A}} s_{\text{C}} + \imath \left(\text{Im} \sqrt{\sigma} \delta_{\text{C}}^{\text{A}} + \text{Im} \alpha s^{\text{A}} s_{\text{C}} \right) \right] \quad (50)$$

$$\left[\text{Re} \sqrt{\sigma} \delta_{\text{D}}^{\text{B}} + \text{Re} \alpha s^{\text{B}} s_{\text{D}} + \imath \left(\text{Im} \sqrt{\sigma} \delta_{\text{D}}^{\text{B}} + \text{Im} \alpha s^{\text{B}} s_{\text{D}} \right) \right] . \quad (51)$$

By using Eq. (50) in the (13) and comparing with (15) we find the following two possibilities.

For $\|\mathbf{s}\|^2 = 0$ we find

$$\alpha = -\imath \frac{\epsilon}{2 \text{Im} \sqrt{\sigma}} \quad \text{and} \quad \text{Re} \sqrt{\sigma} = 0. \quad (52)$$

The deforming matrix is in this case

$$\phi_{\text{C}}^{\text{A}} = \imath \left(\text{Im} \sqrt{\sigma} \delta_{\text{C}}^{\text{A}} - \frac{\epsilon}{2 \text{Im} \sqrt{\sigma}} s^{\text{A}} s_{\text{C}} \right) . \quad (53)$$

For $\|\mathbf{s}\|^2 \neq 0$ we obtain

$$\alpha = \imath \frac{-\text{Im} \sqrt{\sigma} \mp \sqrt{(\text{Im} \sqrt{\sigma})^2 - \epsilon \|\mathbf{s}\|^2}}{\|\mathbf{s}\|^2} \quad (54)$$

$$\text{and} \quad \text{Re} \sqrt{\sigma} = 0 \quad \text{for} \quad \sigma + \epsilon \|\mathbf{s}\|^2 < 0 \quad \text{where} \quad \sigma = -(\text{Im} \sqrt{\sigma})^2.$$

Finally, using (54) in (40) we find the following expression for $\phi_{\text{C}}^{\text{A}}$:

$$\phi_{\text{C}}^{\text{A}} = \imath \left[\text{Im} \sqrt{\sigma} \delta_{\text{C}}^{\text{A}} - \|\mathbf{s}\|^{-2} \left(\text{Im} \sqrt{\sigma} \pm \sqrt{(\text{Im} \sqrt{\sigma})^2 - \epsilon \|\mathbf{s}\|^2} \right) s^{\text{A}} s_{\text{C}} \right] \quad (55)$$

for $\sigma + \epsilon \|\mathbf{s}\|^2 < 0$

Thus, in particular we find

$$\phi_{\text{C}}^{\text{A}} = \imath \left[\text{Im} \sqrt{\sigma} \delta_{\text{C}}^{\text{A}} - \|\mathbf{s}\|^{-2} \left(\text{Im} \sqrt{\sigma} \pm \sqrt{(\text{Im} \sqrt{\sigma})^2 + \|\mathbf{s}\|^2} \right) s^{\text{A}} s_{\text{C}} \right] \quad (56)$$

for $\epsilon = -1$ and $\|\mathbf{s}\|^2 - \sigma > 0$

and

$$\phi_{\text{C}}^{\text{A}} = \imath \left[\text{Im} \sqrt{\sigma} \delta_{\text{C}}^{\text{A}} - \|\mathbf{s}\|^{-2} \left(\text{Im} \sqrt{\sigma} \pm \sqrt{(\text{Im} \sqrt{\sigma})^2 - \|\mathbf{s}\|^2} \right) s^{\text{A}} s_{\text{C}} \right] \quad (57)$$

for $\epsilon = +1$ and $\|\mathbf{s}\|^2 < -\sigma$

While for

$$\sigma + \epsilon \|\mathbf{s}\|^2 > 0 \quad (58)$$

we find:

$$\alpha = \mp \frac{\sqrt{\epsilon \|\mathbf{s}\|^2 - (\text{Im} \sqrt{\sigma})^2}}{\|\mathbf{s}\|^2} - \imath \frac{\text{Im} \sqrt{\sigma}}{\|\mathbf{s}\|^2} \quad \text{and} \quad \text{Re} \sqrt{\sigma} = 0.$$

Finally the matrix ϕ_C^A in this case is

$$\phi_C^A = \mp \frac{\sqrt{\epsilon \|\mathbf{s}\|^2 - (\text{Im} \sqrt{\sigma})^2}}{\|\mathbf{s}\|^2} s^A s_C + \imath \left(\text{Im} \sqrt{\sigma} \delta_C^A - \frac{\text{Im} \sqrt{\sigma}}{\|\mathbf{s}\|^2} s^A s_C \right). \quad (59)$$

Thus, in particular we have:

$$\begin{aligned} \phi_C^A &= \mp \frac{\sqrt{-[\|\mathbf{s}\|^2 + (\text{Im} \sqrt{\sigma})^2]}}{\|\mathbf{s}\|^2} s^A s_C + \imath \left(\text{Im} \sqrt{\sigma} \delta_C^A - \frac{\text{Im} \sqrt{\sigma}}{\|\mathbf{s}\|^2} s^A s_C \right) \\ \text{for } \epsilon &= -1 \quad \text{and} \quad \|\mathbf{s}\|^2 < \sigma \end{aligned} \quad (60)$$

and

$$\begin{aligned} \phi_C^A &= \mp \frac{\sqrt{\|\mathbf{s}\|^2 - (\text{Im} \sqrt{\sigma})^2}}{\|\mathbf{s}\|^2} s^A s_C + \imath \left(\text{Im} \sqrt{\sigma} \delta_C^A - \frac{\text{Im} \sqrt{\sigma}}{\|\mathbf{s}\|^2} s^A s_C \right) \\ \text{for } \epsilon &= +1 \quad \text{and} \quad \|\mathbf{s}\|^2 > -\sigma \end{aligned} \quad (61)$$

We summarize the complex solutions of Eq. (14) for $\sigma < 0$ in Table 2 and for $\sigma > 0$ in Table 3.

$\sigma < 0$	$\epsilon = -1$	$\epsilon = +1$
$\ \mathbf{s}\ ^2 < - \sigma $	$\phi_C^A = \mp \ \mathbf{s}\ ^{-2} \sqrt{-\ \mathbf{s}\ ^2 - (\text{Im} \sqrt{\sigma})^2} s^A s_C + \imath \left(\text{Im} \sqrt{\sigma} \delta_C^A - \frac{\text{Im} \sqrt{\sigma}}{\ \mathbf{s}\ ^2} s^A s_C \right)$	$\phi_C^A = \imath \left[\text{Im} \sqrt{\sigma} \delta_C^A - \ \mathbf{s}\ ^{-2} \left(\text{Im} \sqrt{\sigma} \pm \sqrt{(\text{Im} \sqrt{\sigma})^2 - \ \mathbf{s}\ ^2} \right) s^A s_C \right]$
$\ \mathbf{s}\ ^2 = - \sigma $	$\phi_C^A = \imath \left(\text{Im} \sqrt{\sigma} \delta_C^A + \frac{1}{\text{Im} \sqrt{\sigma}} s^A s_C \right)$	$\phi_C^A = \imath \left[\text{Im} \sqrt{\sigma} \delta_C^A + \frac{1}{\text{Im} \sqrt{\sigma}} (1 \pm \sqrt{2}) s^A s_C \right]$
$\ \mathbf{s}\ ^2 \in]- \sigma , 0[$	$\phi_C^A = \imath \left[\text{Im} \sqrt{\sigma} \delta_C^A - \ \mathbf{s}\ ^{-2} \left(\text{Im} \sqrt{\sigma} \pm \sqrt{(\text{Im} \sqrt{\sigma})^2 + \ \mathbf{s}\ ^2} \right) s^A s_C \right]$	$\phi_C^A = \imath \left[\text{Im} \sqrt{\sigma} \delta_C^A - \ \mathbf{s}\ ^{-2} \left(\text{Im} \sqrt{\sigma} \pm \sqrt{(\text{Im} \sqrt{\sigma})^2 - \ \mathbf{s}\ ^2} \right) s^A s_C \right]$
$\ \mathbf{s}\ ^2 = 0$	$\phi_C^A = \imath \left(\text{Im} \sqrt{\sigma} \delta_C^A + \frac{1}{2 \text{Im} \sqrt{\sigma}} s^A s_C \right)$	$\phi_C^A = \imath \left(\text{Im} \sqrt{\sigma} \delta_C^A - \frac{1}{2 \text{Im} \sqrt{\sigma}} s^A s_C \right)$
$\ \mathbf{s}\ ^2 \in]0, \sigma[$	$\phi_C^A = \imath \left[\text{Im} \sqrt{\sigma} \delta_C^A - \ \mathbf{s}\ ^{-2} \left(\text{Im} \sqrt{\sigma} \pm \sqrt{(\text{Im} \sqrt{\sigma})^2 + \ \mathbf{s}\ ^2} \right) s^A s_C \right]$	$\phi_C^A = \imath \left[\text{Im} \sqrt{\sigma} \delta_C^A - \ \mathbf{s}\ ^{-2} \left(\text{Im} \sqrt{\sigma} \pm \sqrt{(\text{Im} \sqrt{\sigma})^2 - \ \mathbf{s}\ ^2} \right) s^A s_C \right]$
$\ \mathbf{s}\ ^2 = \sigma $	$\phi_C^A = \imath \left[\text{Im} \sqrt{\sigma} \delta_C^A - \frac{1}{\text{Im} \sqrt{\sigma}} (1 \pm \sqrt{2}) s^A s_C \right]$	$\phi_C^A = \imath \left[\text{Im} \sqrt{\sigma} \delta_C^A - \frac{1}{\text{Im} \sqrt{\sigma}} s^A s_C \right]$
$\ \mathbf{s}\ ^2 > \sigma $	$\phi_C^A = \imath \left[\text{Im} \sqrt{\sigma} \delta_C^A - \ \mathbf{s}\ ^{-2} \left(\text{Im} \sqrt{\sigma} \pm \sqrt{(\text{Im} \sqrt{\sigma})^2 + \ \mathbf{s}\ ^2} \right) s^A s_C \right]$	$\phi_C^A = \mp \ \mathbf{s}\ ^{-2} \sqrt{\ \mathbf{s}\ ^2 - (\text{Im} \sqrt{\sigma})^2} s^A s_C + \imath \left(\text{Im} \sqrt{\sigma} \delta_C^A - \frac{\text{Im} \sqrt{\sigma}}{\ \mathbf{s}\ ^2} s^A s_C \right)$

Table 2 Deforming matrices for three-dimensional metrics. 2° case: Complex solutions with $\sigma < 0$. In this case we find $\text{Re} \sqrt{\sigma} = 0$ and $|\sigma| = (\text{Im} \sqrt{\sigma})^2$.

5 The causal structure

In this section we investigate the casual structure of a deformed manifold compared with the original one. In order to start our considerations, we show how, for h , σ and \mathbf{s} fixed, the norm of tangent vectors on the manifold M changes for deformation of the metric tensors. Thus, we consider the norm

$\sigma > 0$	$\epsilon = -1$	$\epsilon = +1$
$\ \mathbf{s}\ ^2 < -\sigma$	✓	$\phi_C^A = \sqrt{\sigma} \delta_C^A - \frac{\sqrt{\sigma}}{\ \mathbf{s}\ ^2} s^A s_C \mp \frac{1}{\ \mathbf{s}\ ^2} \sqrt{-(\sigma + \ \mathbf{s}\ ^2)} s^A s_C$
$\ \mathbf{s}\ ^2 = -\sigma$	✓	$\phi_C^A = \sqrt{\sigma} \delta_C^A + \frac{1}{\sqrt{\sigma}} s^A s_C$
$\ \mathbf{s}\ ^2 = 0$	$\phi_C^A = \sqrt{\sigma} - \frac{1}{2\sqrt{\sigma}} s^A s_C$	$\phi_C^A = \sqrt{\sigma} + \frac{1}{2\sqrt{\sigma}} s^A s_C$
$\ \mathbf{s}\ ^2 = \sigma$	$\phi_C^A = \sqrt{\sigma} \delta_C^A - \frac{1}{\sqrt{\sigma}} s^A s_C$	✓
$\ \mathbf{s}\ ^2 > \sigma$	$\phi_C^A = \sqrt{\sigma} \delta_C^A - \frac{\sqrt{\sigma}}{\ \mathbf{s}\ ^2} s^A s_C \mp \frac{1}{\ \mathbf{s}\ ^2} \sqrt{-(\sigma - \ \mathbf{s}\ ^2)} s^A s_C$	✓

Table 3 Deforming matrices for three-dimensional metrics. 2° case: Complex solutions with $\sigma > 0$.

$\|\mathbf{A}\|_g^2$ of a vector \mathbf{A} , defined by the deformed metric g and the $\|\mathbf{A}\|_h^2$ defined from the metric h as follows:

$$\|\mathbf{A}\|_g^2 \equiv g_{\mu\nu} A^\mu A^\nu, \quad \|\mathbf{A}\|_h^2 \equiv h_{\mu\nu} A^\mu A^\nu. \quad (62)$$

In the following, for simplicity we will use the notation

$$(sA)^2 \equiv s_\mu A^\mu s_\nu A^\nu. \quad (63)$$

Therefore, writing the norms as:

$$g_{\mu\nu} A^\mu A^\nu = (\sigma h_{\mu\nu} + \epsilon s_\mu s_\nu) A^\mu A^\nu \quad (64)$$

we have

$$(\sigma h_{\mu\nu} + \epsilon s_\mu s_\nu) A^\mu A^\nu = \sigma h_{\mu\nu} A^\mu A^\nu + \epsilon s_\mu A^\mu s_\nu A^\nu, \quad (65)$$

$$\|\mathbf{A}\|_g^2 = \sigma \|\mathbf{A}\|_h^2 + \epsilon (sA)^2. \quad (66)$$

Moreover, at (66) a scalar relation $\Psi(\sigma, \|\mathbf{s}\|) = 0$ between σ and \mathbf{s} should be imposed. Below we describe the various cases we find from this decomposition. We can distinguish the cases in which the casual structure remains unchanged and the cases in which it is locally and globally changed.

1° caso: $\mathbf{s} = 0$. In this first case transformation (11) is a conformal deformation, thus

$$\|\mathbf{A}\|_g^2 = \sigma \|\mathbf{A}\|_h^2.$$

We note that if σ is a generic function, where this is positive the casual structure is preserved, while where it is negative the casual structure is changed, and if the function is continue, it will be zero in some regions, thus there will be a singularity. This pathologic situation could be avoided if we consider only strictly positive scalar fields σ . In this case the casual structure will be preserved in the entire spacetime.

2° case: $\mathbf{s} \neq \mathbf{0}$ Let us split this second case into two subcases.

Let be $\mathbf{A} \neq \mathbf{0}$ a vector, if $s_\mu A^\mu = 0$ then $\|\mathbf{A}\|_g^2 \equiv \sigma \|\mathbf{A}\|_h^2$, or we have $s_\mu A^\mu \neq 0$.

If \mathbf{A} and \mathbf{s} are real vectors we have

$$(\mathbf{sA})^2 > 0. \quad (67)$$

From the relation

$$\|\mathbf{A}\|_g^2 = \sigma \|\mathbf{A}\|_h^2 + \epsilon (\mathbf{sA})^2 \quad (68)$$

we have the following possibilities

1. If $\sigma \|\mathbf{A}\|_h^2 > (\mathbf{sA})^2$ then

$$\|\mathbf{A}\|_g^2 > 0 \quad \text{for } \epsilon = \pm 1, \quad (69)$$

2. If $\sigma \|\mathbf{A}\|_h^2 = (\mathbf{sA})^2$ we have:

$$\|\mathbf{A}\|_g^2 > 0 \quad \text{for } \epsilon = +1, \quad (70a)$$

$$\|\mathbf{A}\|_g^2 = 0 \quad \text{for } \epsilon = -1, \quad (70b)$$

3. If $\sigma \|\mathbf{A}\|_h^2 < (\mathbf{sA})^2$ then we have the following three possibilities:

If $0 < \sigma \|\mathbf{A}\|_h^2 < (\mathbf{sA})^2$

$$\|\mathbf{A}\|_g^2 > 0 \quad \text{for } \epsilon = +1, \quad (71a)$$

$$\|\mathbf{A}\|_g^2 < 0 \quad \text{for } \epsilon = -1. \quad (71b)$$

If $\sigma \|\mathbf{A}\|_h^2 = 0$ instead:

$$\|\mathbf{A}\|_g^2 > 0 \quad \text{for } \epsilon = +1, \quad (72a)$$

$$\|\mathbf{A}\|_g^2 < 0 \quad \text{for } \epsilon = -1. \quad (72b)$$

If $\|\mathbf{A}\|_{\sigma h}^2 < 0$ we finally find the results listed in Table 4, where $|\sigma \|\mathbf{A}\|_h^2|$ is the module of $\sigma \|\mathbf{A}\|_h^2$. In the Table 5 we report the results obtained

$\sigma \ \mathbf{A}\ _h^2 < 0$	$\epsilon = -1$	$\epsilon = +1$
$ \sigma \ \mathbf{A}\ _h^2 > (\mathbf{sA})^2$	$\ \mathbf{A}\ _g^2 < 0$	$\ \mathbf{A}\ _g^2 < 0$
$ \sigma \ \mathbf{A}\ _h^2 < (\mathbf{sA})^2$	$\ \mathbf{A}\ _g^2 < 0$	$\ \mathbf{A}\ _g^2 > 0$
$ \sigma \ \mathbf{A}\ _h^2 = (\mathbf{sA})^2$	$\ \mathbf{A}\ _g^2 < 0$	$\ \mathbf{A}\ _g^2 = 0$

Table 4 Casual structure of a deformed manifold: case $\|\mathbf{A}\|_{\sigma h}^2 < 0$.

for $(\mathbf{sA})^2 > 0$.

We note that the case $(\mathbf{sA})^2 < 0$ should be avoided because Eq. (11) is proved only in the analytical case, where all the data are real analytic functions, however since the case of complex \mathbf{s} vector occurs from deformation (6) parameterized by scalar fields, this case is also studied. We summarize the results for the $(\mathbf{sA})^2 < 0$ case in Table 6.

$(\mathbf{sA})^2 > 0$	$\epsilon = -1$	$\epsilon = +1$
$\sigma \ \mathbf{A}\ _h^2 > (\mathbf{sA})^2$	$\ \mathbf{A}\ _g^2 > 0$	$\ \mathbf{A}\ _g^2 > 0$
$\sigma \ \mathbf{A}\ _h^2 = (\mathbf{sA})^2$	$\ \mathbf{A}\ _g^2 = 0$	$\ \mathbf{A}\ _g^2 = 2\sigma \ \mathbf{A}\ _h^2$
$\sigma \ \mathbf{A}\ _h^2 \in] -(\mathbf{sA})^2, (\mathbf{sA})^2 [$	$\ \mathbf{A}\ _g^2 < 0$	$\ \mathbf{A}\ _g^2 > 0$
$\sigma \ \mathbf{A}\ _h^2 = 0$	$\ \mathbf{A}\ _g^2 = -(\mathbf{sA})^2$	$\ \mathbf{A}\ _g^2 = (\mathbf{sA})^2$
$\sigma \ \mathbf{A}\ _h^2 = -(\mathbf{sA})^2$	$\ \mathbf{A}\ _g^2 = -2\sigma \ \mathbf{A}\ _h^2$	$\ \mathbf{A}\ _g^2 = 0$
$\sigma \ \mathbf{A}\ _h^2 < -(\mathbf{sA})^2$	$\ \mathbf{A}\ _g^2 < 0$	$\ \mathbf{A}\ _g^2 < 0$

Table 5 Casual structure of a deformed manifold. Case $(\mathbf{sA})^2 > 0$

$(\mathbf{sA})^2 < 0$	$\epsilon = -1$	$\epsilon = +1$
$\sigma \ \mathbf{A}\ _h^2 < (\mathbf{sA})^2$	$\ \mathbf{A}\ _g^2 < 0$	$\ \mathbf{A}\ _g^2 < 0$
$\sigma \ \mathbf{A}\ _h^2 = (\mathbf{sA})^2$	$\ \mathbf{A}\ _g^2 = 0$	$\ \mathbf{A}\ _g^2 = 2\sigma \ \mathbf{A}\ _h^2$
$\sigma \ \mathbf{A}\ _h^2 \in](\mathbf{sA})^2, -(\mathbf{sA})^2 [$	$\ \mathbf{A}\ _g^2 > 0$	$\ \mathbf{A}\ _g^2 < 0$
$\sigma \ \mathbf{A}\ _h^2 = 0$	$\ \mathbf{A}\ _g^2 = -(\mathbf{sA})^2$	$\ \mathbf{A}\ _g^2 = (\mathbf{sA})^2$
$\sigma \ \mathbf{A}\ _h^2 = -(\mathbf{sA})^2$	$\ \mathbf{A}\ _g^2 = -2\sigma \ \mathbf{A}\ _h^2$	$\ \mathbf{A}\ _g^2 = 0$
$\sigma \ \mathbf{A}\ _h^2 > -(\mathbf{sA})^2$	$\ \mathbf{A}\ _g^2 > 0$	$\ \mathbf{A}\ _g^2 > 0$

Table 6 Casual structure of a deformed manifold. Case $(\mathbf{sA})^2 < 0$

6 Some deformed three-dimensional metrics

In this section we investigate the local deformations of some three-dimensional Riemannian manifolds into flat metrics. We start by applying the result (15) to the particular simple cases of constant curvature metrics studied in [1]: the metric of a three-dimensional sphere, and the metric of a Kerr space. Then, by using the results listed in Table 1 and 2, we construct the deforming matrices associated to these deformed metrics.

6.1 The three-dimensional sphere

We consider the three-dimensional sphere S^3 . The metric for the three-manifold in Schwarzschild spacetime, is²:

$$\hat{g} = \left(1 - \frac{2m}{r}\right)^{-1} dr \otimes dr + r^2 (d\theta \otimes d\theta + \sin^2 \theta d\phi \otimes d\phi) \quad (73)$$

² In units of $c = G = 1$.

in the polar coordinate (r, θ, ϕ) where m is the body mass and the metric is considered in the region $r > 2m$, [26, 27].

Metric (73) can be locally deformed into a flat metric in several ways. For example in [1] the metric \hat{g} is obtained by the deformation of a metric \tilde{g} by the (11) as:

$$\hat{g} = \tilde{g} + \mathbf{s} \otimes \mathbf{s}, \quad (74)$$

where $\sigma = 1$, $\epsilon = +1$ and $\mathbf{s} = \sqrt{k^{-1} - 1} dr$ where $k \equiv (1 - \frac{2m}{r})$ and

$$\tilde{g} = dr \otimes dr + r^2 (d\theta \otimes d\theta + \sin^2 \theta d\phi \otimes d\phi). \quad (75)$$

We can write \tilde{g} as

$$\tilde{g} = \omega^{\hat{r}} \otimes \omega^{\hat{r}} + \omega^{\hat{\theta}} \otimes \omega^{\hat{\theta}} + \omega^{\hat{\phi}} \otimes \omega^{\hat{\phi}}, \quad (76)$$

in the triad ω_a^A

$$\omega^{\hat{r}} = dr, \quad \omega^{\hat{\theta}} = r d\theta, \quad \omega^{\hat{\phi}} = r \sin \theta d\phi \quad (77)$$

while the metric \hat{g} reads

$$\hat{g} = \frac{1}{k} \omega^{\hat{r}} \otimes \omega^{\hat{r}} + \omega^{\hat{\theta}} \otimes \omega^{\hat{\theta}} + \omega^{\hat{\phi}} \otimes \omega^{\hat{\phi}} \quad (78)$$

The deforming matrix ϕ_C^A associated to the deformation (74) has the form (16), or

$$\phi_C^A = \sqrt{\sigma} \delta_C^A + \alpha s^A s_C, \quad (79)$$

where in this case $\sigma = 1$ and $\|\mathbf{s}\|_g^2 \equiv (1/k - 1) > 0$.

In particular equation (15) has the following solutions:

$$\phi_C^A = \sqrt{\sigma} \delta_C^A + \frac{-\sqrt{\sigma} \mp \sqrt{\sigma + \|\mathbf{s}\|^2}}{\|\mathbf{s}\|^2} \mathbf{s}^A \mathbf{s}_C, \quad \text{for } \|\mathbf{s}\|_g^2 \in]0, 1[\quad (80)$$

and substituting in the (80) $\|\mathbf{s}\|_g^2$ and σ we have:

$$\phi_C^A = \delta_C^A - \left(1 \pm \frac{1}{\sqrt{k}}\right) \frac{1}{(\frac{1}{k} - 1)} \mathbf{s}^A \mathbf{s}_C, \quad \text{for } k \in]1/2, 1[\quad (81)$$

or also

$$\phi_C^A = \delta_C^A - \frac{r}{2m} \sqrt{1 - \frac{2m}{r}} \left(\sqrt{1 - \frac{2m}{r}} \pm 1 \right) \mathbf{s}^A \mathbf{s}_C, \quad \text{for } r > 4m \quad (82)$$

In particular for $k = 1/2$:

$$\phi_C^A = \sqrt{\sigma} \delta_C^A + (-1 \pm \sqrt{2}) \mathbf{s}^A \mathbf{s}_C, \quad \text{for } \|\mathbf{s}\|_g^2 \equiv 1 \quad \text{or } r = 4m \quad (83)$$

Finally, we have:

$$\phi_C^A = \sqrt{\sigma} \delta_C^A + \frac{-\sqrt{\sigma} \mp \sqrt{\sigma + \|\mathbf{s}\|^2}}{\|\mathbf{s}\|^2} \mathbf{s}^A \mathbf{s}_C, \quad \text{for } \|\mathbf{s}\|_g^2 \in]1, +\infty[\quad (84)$$

Substituting $\|\mathbf{s}\|_{\tilde{g}}^2$ and σ in the (84) we have:

$$\phi_{\mathbf{c}}^{\mathbf{A}} = \delta_{\mathbf{c}}^{\mathbf{A}} - \left(1 \pm \frac{1}{\sqrt{k}}\right) \frac{1}{\left(\frac{1}{k} - 1\right)} \mathbf{s}^{\mathbf{A}} \mathbf{s}_{\mathbf{c}}, \quad \text{for } k < 1/2 \quad (85)$$

or also, for $r \in]2m, 4m[$

$$\phi_{\mathbf{c}}^{\mathbf{A}} = \delta_{\mathbf{c}}^{\mathbf{A}} - \frac{r}{2m} \sqrt{1 - \frac{2m}{r}} \left(\sqrt{1 - \frac{2m}{r}} \pm 1 \right) \mathbf{s}^{\mathbf{A}} \mathbf{s}_{\mathbf{c}}, \quad (86)$$

We note that, the solution (84) matches (80).

Alternatively it is possible to deform the metric (73) in a flat metric by changing the coordinate r into the coordinate R defined as:

$$R \equiv \frac{1}{2} \left(r\sqrt{k} + r - m \right), \quad \text{where } r = R \left(1 + \frac{m}{2R} \right)^2. \quad (87)$$

We can write \tilde{g} as

$$\tilde{g} = \omega^{\hat{r}} \otimes \omega^{\hat{r}} + \omega^{\hat{\theta}} \otimes \omega^{\hat{\theta}} + \omega^{\hat{\phi}} \otimes \omega^{\hat{\phi}}, \quad (88)$$

in the triad $\omega_a^{\mathbf{A}}$ defined as

$$\omega^{\hat{r}} = \left(1 - \frac{m^2}{4R^2} \right)^{-1} dr, \quad \omega^{\hat{\theta}} = R d\theta, \quad \omega^{\hat{\phi}} = R \sin \theta d\phi \quad (89)$$

while the metric \hat{g} is simplify a conformal deformation of \tilde{g}

$$\hat{g} = \sigma \tilde{g}, \quad \sigma \equiv \left(1 + \frac{m}{2R} \right)^4 \quad (90)$$

and \mathbf{s} is the zero-vector. In this case the deforming matrix is

$$\phi_{\mathbf{c}}^{\mathbf{A}} = \sqrt{\sigma} \Lambda_{\mathbf{c}}^{\mathbf{A}}. \quad (91)$$

6.2 Three-dimensional Kerr space

Consider the following Kerr metric in the Boyer-Lindquist coordinates:

$$g = -dt^2 + \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2 + (r^2 + a^2) \sin^2 \theta d\phi^2 + \frac{2m}{\rho^2} r (dt - a \sin^2 \theta d\phi)^2, \quad (92)$$

where

$$\rho^2 \equiv r^2 + a^2 \cos^2 \theta, \quad \Delta \equiv r^2 - 2mr + a^2. \quad (93)$$

and $\rho^2 - 2mr > 0$; here m is a mass parameter and the specific angular momentum is given as $a = J/m$, where J is the total angular momentum of the gravitational source, we consider the Kerr black hole spacetime defined by $a \in]0, m[$, the extreme black hole source is for $a = m$, the limiting case $a = 0$ is the Schwarzschild solution [26, 27, 28, 29, 30, 31, 32].

We consider the stationary metric \hat{g} defined as

$$\hat{g} \equiv \frac{\rho^2}{r^2} \left(\frac{r^2}{\Delta} dr \otimes dr + r^2 d\theta \otimes d\theta + \frac{\mathcal{A}}{\rho^4} r^2 \sin^2 \theta d\phi \otimes d\phi \right), \quad (94)$$

where $\mathcal{A} \equiv (r^2 + a^2)^2 - \Delta a^2 \sin^2 \theta$. First we change the coordinate r into the coordinate $R \equiv R_+$ defined as follows

$$R_{\pm} \equiv \frac{1}{2} \left(r - m \pm \sqrt{\Delta} \right), \quad \text{where} \quad r = R \left[\left(1 + \frac{m}{2R} \right)^2 - \frac{a^2}{4R^2} \right]. \quad (95)$$

for $a = 0$ this reduces to the isotropic radius for the Schwarzschild metric. We can introduce the metric tensor \tilde{g}

$$\tilde{g} = \omega^{\hat{r}} \otimes \omega^{\hat{r}} + \omega^{\hat{\theta}} \otimes \omega^{\hat{\theta}} + \omega^{\hat{\phi}} \otimes \omega^{\hat{\phi}}, \quad (96)$$

in the triad $\omega_a^{\hat{A}}$, defined as

$$\omega^{\hat{r}} = \left(1 - \frac{m^2}{4R^2} + \frac{a^2}{4R^2} \right)^{-1} dr, \quad \omega^{\hat{\theta}} = R d\theta, \quad \omega^{\hat{\phi}} = \sqrt{\frac{\mathcal{A}R^2}{\rho^4} - \frac{a^2 m^2}{(\rho^2 - 2mr)}} \sin \theta d\phi \quad (97)$$

while the metric \hat{g} reads

$$\hat{g} = \sigma \tilde{g} + \mathbf{s} \otimes \mathbf{s} \quad (98)$$

where $\epsilon = +1$ and

$$\sigma \equiv \frac{\rho^2}{R^2}, \quad \mathbf{s} \equiv \frac{\rho a \sin \theta}{R \sqrt{\rho^2 - 2mr}} \omega^{\hat{\phi}}, \quad (99)$$

and

$$\|\mathbf{s}\|_{\tilde{g}}^2 = \sigma \left(\frac{a^2 \sin^2 \theta}{\rho^2 - 2mr} \right) > 0. \quad (100)$$

The deforming matrix $\phi_{\text{C}}^{\text{A}}$ associated to the deformation (98) has the form (16). In particular Eq.(15) has the following solutions:

$$\phi_{\text{C}}^{\text{A}} = \sqrt{\sigma} \delta_{\text{C}}^{\text{A}} + \frac{-\sqrt{\sigma} \mp \sqrt{\sigma + \|\mathbf{s}\|^2}}{\|\mathbf{s}\|^2} \mathbf{s}^{\text{A}} \mathbf{s}_{\text{C}}, \quad \text{for} \quad \|\mathbf{s}\|_{\tilde{g}}^2 \in]0, \sigma[\quad (101)$$

and substituting in the (101) the explicit forms (99) and (100) for $\|\mathbf{s}\|_{\tilde{g}}^2$ and σ respectively we obtain:

$$\phi_{\text{C}}^{\text{A}} = \frac{\rho}{R} \delta_{\text{C}}^{\text{A}} - \frac{R}{\rho} \frac{\sqrt{\rho^2 - 2mr}}{a^2 \sin^2 \theta} \left(\sqrt{\rho^2 - 2mr} \pm \sqrt{\Delta} \right) \mathbf{s}^{\text{A}} \mathbf{s}_{\text{C}}, \quad (102)$$

for

$$\rho^2 - 2mr > a^2 \sin^2 \theta. \quad (103)$$

In particular:

$$\phi^A_C = \frac{\rho}{R} \delta^A_C + \frac{R}{2\rho} \mathbf{s}^A \mathbf{s}_C, \text{ for } \|\mathbf{s}\|_g^2 \equiv 0 \quad \text{or also} \quad a \sin \theta = 0 \quad (104)$$

meanwhile:

$$\begin{aligned} \phi^A_C &= \frac{\rho}{R} \delta^A_C - \frac{R}{\rho} (1 \pm \sqrt{2}) \mathbf{s}^A \mathbf{s}_C, \\ \text{for } \|\mathbf{s}\|_g^2 &\equiv \sigma \quad \text{or also} \quad \rho^2 - 2mr = a^2 \sin^2 \theta. \end{aligned} \quad (105)$$

Finally, we have:

$$\phi^A_C = \sqrt{\sigma} \delta^A_C + \frac{-\sqrt{\sigma} \mp \sqrt{\sigma + \|\mathbf{s}\|^2}}{\|\mathbf{s}\|^2} \mathbf{s}^A \mathbf{s}_C, \text{ for } \|\mathbf{s}\|_g^2 \in]\sigma, +\infty[\quad (106)$$

and using Eqs. (99) and (100) in the (106) we have:

$$\begin{aligned} \phi^A_C &= \frac{\rho}{R} \delta^A_C - \frac{R}{\rho} \frac{\sqrt{\rho^2 - 2mr}}{a^2 \sin^2 \theta} \left(\sqrt{\rho^2 - 2mr} \pm \sqrt{\Delta} \right) \mathbf{s}^A \mathbf{s}_C, \\ \text{for } 0 < \rho^2 - 2mr &< a^2 \sin^2 \theta. \end{aligned} \quad (107)$$

We note that the solution (106) matches (101).

7 Discussion and conclusions

In this work we studied deformations of three dimensional metrics. We related two different procedures to obtain a deformed metric tensor of a three dimensional manifold. We started considering the procedure outlined in [16] where a deformation of three-dimensional metric tensor h was obtained by first multiplying it by a conformal transformation with a scalar field σ and then summing it with the tensor product $\mathbf{s} \otimes \mathbf{s}$ of the differential 1-form \mathbf{s} . The deformation so defined is applied to attain a constant curvature Riemannian metric. However, Theorem 1 is proved only locally, in the neighborhood of the Cauchy surface, and in the analytical case where all the data are real analytic functions. Then we introduced the metric deformations according to [22]. They seem to be more general because they allow to define a transformation between any two metrics by means of generic, not necessarily real, scalar fields. By writing the algebraic equations which identified the metric transformations of [16] with the single components of the deforming matrices, we found the explicit expressions for the generic deforming matrices. We discussed separately the real and complex solutions and provided a classification of various deforming matrices according to space-time properties of σ and of \mathbf{s} . Since the deformations obtained are not just conformal transformations, we expect that the casual structure of the deformed manifold is modified and we found the conditions under which it is preserved.

Finally, as an application, we presented in terms of a flat metric, to the geometry of a the three-dimensional sphere and of the spatial sector of a Kerr space-time.

While the work of Coll et al. gives an interesting approach to deforming metrics it seems to be an extension of the family of the Kerr-Schild metrics. The deformation by scalar matrices seems to be more general. And one of its main advantages is that it is covariant with respect to coordinate transformations, the price payed for this is that the deforming matrices are not uniquely defined.

We expect deformations to be amenable to different physical applications in general relativity and extended theories of gravity. For example they can be applied to cast the solutions of the gravitational equations in a suitable manner according to astrophysical observations or for studying numerical implementations. Furthermore it is possible to use the space-time metric deformations as geometric perturbations.

These aspects of space-time deformations and other considerations on the possible associated phenomenology, as gravitational lensing and redshift, have been addressed in [23, 25], where it is discussed in particular the possibility to constraint the deforming scalar fields. In this respect, these considerations can be directions this work could take in the future from the results in Section (4) and to be investigated elsewhere.

The present analysis has been restricted to the case of three dimensional metrics, but a more general version of these results for arbitrary dimensions is also possible.

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